

ROTATING 5D-KALUZA-KLEIN SPACE-TIMES FROM INVARIANT TRANSFORMATIONS

Tonatiuh Matos^a, Darío Núñez^b, Gabino Estevez^a, Maribel Rios^c

^aDepartamento de Física,

Centro de Investigación y Estudios Avanzados del I. P. N.,

A. P. 14-700, 07000 México, D.F., MEXICO

^bInstituto de Ciencias Nucleares,

Universidad Nacional Autónoma de México

A. P. 70-543, 04510 México, D. F., MEXICO

^c Instituto de Física y Matemáticas

Universidad Michoacana de San Nicolás de Hidalgo

Apdo. Postal 2-82, 58040 Morelia, Michoacán, México

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Abstract

Using invariant transformations of the five-dimensional Kaluza-Klein (KK) field equations, we find a series of formulae to derive axial symmetric stationary exact solutions of the KK theory starting from static ones. The procedure presented in this work allows to derive new exact solutions up to very simple integrations. Among other results, we find exact rotating solutions containing magnetic monopoles, dipoles, quadripoles, etc., coupled to scalar and to gravitational multipole fields.

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1 Introduction

In recent years, dilaton fields have been proposed as a strong candidate for describing dark matter. At a cosmological level it has been used to explain the Large Scale Structure of the Universe [1][2]. At a galactic level, scalar fields seem to play a crucial role in explaining the curves of rotational velocities vs. radius, observed in all the galaxies [3][4], and it is thought that it will also play an important role in several physical phenomena at a local level [5][6], *i.e.*, in the realm of compact objects.

From a theoretical point of view, dilaton fields coupled to Einstein-Maxwell fields, naturally appear in the low energy limit of string theory, and as a result of a dimensional reduction of the Kaluza-Klein Lagrangian. Therefore, the study of the Einstein-Maxwell-Dilaton theory is of importance to investigate the properties of compact objects involving these fields and for the understanding of more general theories.

On the other hand, if the scalar fields are so important in physics, why they have not been yet detected? As we just mentioned, in several models they play an important role, but due to the fact that they interact very weakly with matter [6, 7], in most of the observational tests the results at most just can not exclude them. It is expected that the scalar fields will have an important measurable signature of their presence in regions with strong gravitational fields [8]. Thus, it is necessary to have exact analytical solutions to the Einstein-Maxwell-Dilaton theory, not only perturbative solutions, and then compare with the observations the predictions made using those exact solutions. The problem with this approach, is that the field equations are very complicated to be solved exactly and one must recur to mathematical methods which usually prove to be very cumbersome. In this work we want to give some simple formulas which allow us to derive exact rotating dilatons solutions, starting from static ones, and which avoids many of the mathematical difficulties usually encountered in deriving exact solutions from seed ones. In this work we derive three expressions which can be used to generate families of solutions, starting from known seed ones. Out of these expressions, only one has been previously obtained in reference [9]

In order to do so, let us start from the Lagrangian

$$\mathcal{L} = \sqrt{-g}[-R + 2(\nabla\phi)^2 + e^{-2\alpha\phi}F^2] \quad (1)$$

This Lagrangian contains very interesting limits. For $\alpha^2 = 3$ Lagrangian (1)

contains the Kaluza-Klein theory; for $\alpha^2 = 1$, equation (1) represents the effective Lagrangian for the low energy limit of super-strings theory; finally, equation (1) contains the Einstein-Maxwell theory with a minimally couple scalar field for $\alpha^2 = 0$. This Lagrangian is also very convenient because after a conformal transformation of the metric, one can obtain a equivalent Lagrangian for an almost arbitrary scalar-tensor theory of gravity [6](with a non trivial electromagnetic-scalar interaction which can be avoided setting $F^2 = 0$). The field equations derived from Lagrangian (1) are give by

$$\begin{aligned}\nabla_\mu(e^{-2\alpha\phi})F^{\mu\nu} &= 0; \\ \nabla^2\phi + \frac{\alpha}{2}e^{-2\alpha\phi}F^2 &= 0; \\ R_{\mu\nu} &= 2\nabla_\mu\nabla_\nu\phi + 2e^{-2\alpha\phi}(F_{\mu\rho}F_\nu^\rho - \frac{1}{2}g_{\mu\nu}e^{-2\alpha\phi}F^2).\end{aligned}\quad (2)$$

There exist several exact solutions of equations (2) (see reference [10] for solutions with $\alpha^2 = 3$ and [11] for their generalization to α arbitrary). Some of them could be models for the exterior space-time of an astrophysical compact object [12, 7] or of a black hole with a scalar field interaction [13, 14, 11]. Let us give two examples of such metrics. The first space-time we deal with behaves gravitationally like the Schwarzschild solution for $\alpha \neq 0$ and contains an arbitrary magnetic field. This metric reads [10]

$$\begin{aligned}ds^2 &= e^{2k_s}g^\gamma \frac{dr^2}{1 - \frac{2m}{r}} + g^\gamma r^2(e^{2k_s}d\theta^2 + \sin^2\theta d\varphi^2) - \frac{1 - \frac{2m}{r}}{g^\gamma} dt^2 \\ A_{03,z} &= Q\rho\tau_{,z} \quad , \quad A_{03,\bar{z}} = -Q\rho\tau_{,\bar{z}} \quad , \quad e^{-2\alpha\phi_0} = \frac{k_1^2}{(1 - \frac{2m}{r})g^\beta}\end{aligned}\quad (3)$$

where a subindex 0 stands for a seed solution and

$$g = a_1\tau + 1, \quad \text{and} \quad e^{2k_s} = \left(1 + \frac{m^2 \sin^2\theta}{r^2(1 - \frac{2m}{r})}\right)^{-1/\alpha^2},$$

In this work we use the coordinates $z = \rho + i\zeta = \sqrt{r^2 - 2mr} \sin\theta + i(r - m)\cos\theta$. $\mathbf{A}_0 = A_{0\mu}dx^\mu$, (with $\mu = 1\dots 4$), is the electromagnetic four potential, m the mass parameter, $\gamma = 2/(1 + \alpha^2)$, $\beta = 2\alpha^2/(1 + \alpha^2)$; Q and a_1 are constants related by

$$2\gamma a_1^2 - k_1^2 Q^2 = 0$$

Solution (3) can be interpreted as a magnetized Schwarzschild solution in dilaton gravity for $\alpha \neq 0$. For $\alpha = 0$ the construction of dipoles is different and the form of the metric is not similar to the Schwarzschild solution any more [12]. In what follows, we will assume $\alpha \neq 0$. The function $\tau = \tau(\rho, \zeta)$ is a harmonic parameter in a two dimensional flat space, *i.e.*, it is a solution of the Laplace equation

$$\frac{1}{2\rho}[(\rho\tau_{,z})_{,\bar{z}} + (\rho\tau_{,\bar{z}})_{,z}] = \tau_{\rho\rho} + \frac{1}{\rho}\tau_{,\rho} + \tau_{\zeta\zeta} = 0. \quad (4)$$

This metric represents the exterior field of a gravitational object with an arbitrary magnetic field coupled to a scalar field. The metric is singular for $r = 2m$ and for an interior radius determined by the magnetic field. For a pulsar with magnetic and scalar fields in the region where $r > 2m$, this metric could be an static model of an astrophysical object with magnetic and scalar fields, metric (3) is always regular in that region.

The second metric we will deal with is given by [11]

$$ds^2 = \frac{1}{f_0} \left[e^{2k_0} (d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2 \right] - f_0 dt^2, \quad (5)$$

where

$$\begin{aligned} f_0 &= \frac{e^\lambda}{(a_1\Sigma_1 + a_2\Sigma_2)^\gamma}, \\ e^{-2\alpha\phi_0} &= \kappa_0^2 = \kappa_1^2 (a_1\Sigma_1 + a_2\Sigma_2)^\beta e^{\lambda - \tau_0\tau}, \\ 2A_{04} &= \psi_0 = \frac{a_3\Sigma_1 + a_4\Sigma_2}{a_1\Sigma_1 + a_2\Sigma_2}, \end{aligned} \quad (6)$$

where a_1, \dots, κ_1 , and τ_0 are constants and β, γ are again functions of α defined as $\gamma = 2/(1 + \alpha^2)$, $\beta = 2\alpha^2/(1 + \alpha^2)$; $\tau = \tau(\rho, \zeta)$ and $\lambda = \lambda(\rho, \zeta)$ are harmonic functions which satisfy again the Laplace equation (observe that we have defined a new parameter λ with respect to the one defined in [11]). Σ_1 and Σ_2 are functions given in terms of τ and the equation (6) contains two subclasses determined by the functions Σ_1 and Σ_2 . For the first subclass we have $\tau_0 = 0$ and

$$\Sigma_1 = \tau \quad \Sigma_2 = 1 \quad (7)$$

with the relation between the constants

$$4a_1^2 - \kappa_1^2(1 + \alpha^2)(a_1a_4 - a_2a_3)^2 = 0. \quad (8)$$

For the second subclass we have

$$\Sigma_1 = e^{q_1\tau} \quad \Sigma_2 = e^{q_2\tau}, \quad (9)$$

where q_1, q_2 are constants and τ_0 satisfy the relation $\tau_0 = q_1 + q_2$; the condition for the constants in this case is given by:

$$4a_1a_2 + \kappa_1^2(1 + \alpha^2)(a_1a_4 - a_2a_3)^2 = 0. \quad (10)$$

One of the most interesting solution contained in this class is the Gibbons-Maeda [15] black-hole

$$ds^2 = g^\gamma \frac{dr^2}{1 - \frac{2m}{r}} + g^\gamma r^2(e^{2k_s} d\theta^2 + \sin^2 \theta d\varphi^2) - \frac{1 - \frac{2m}{r}}{g^\gamma} dt^2 \quad (11)$$

where

$$\begin{aligned} e^{2\alpha\phi_0} &= \left(1 + \frac{r_-}{r}\right)^{\frac{-2\alpha^2}{1+\alpha^2}} \\ g &= \left(1 + \frac{r_-}{r}\right); \\ r_- - r_+ &= -2m \end{aligned}$$

(we have used $r \rightarrow r + r_-$ from the original solution). This metric could represent a static charged black hole containing a scalar field ϕ_0 (for a study of this metric see [13, 14]). Observe that in both metrics (3) and (11), the space-time is qualitatively different only for $\alpha = 0$, but for $\alpha \neq 0$ the qualitative behavior is very similar for any α and many of the main features of the metrics can be obtained for a specific α .

On the other hand, real astrophysical objects rotate. For an object like a pulsar, taking into account the rotation is very important in order to understand its space-time configuration. Therefore, if we want to model an astrophysical object with scalar field, we must find the corresponding rotating metrics of (3) and (11) to obtain the solution which we want to use for modeling them.

Using the potential space formalism for 5D gravity introduced by Neugebauer [16], we were able to find a set of formulas valid for $\alpha^2 = 3$ in order to obtain rotating exact solutions from a static one, and without making any complicated integration. We will introduce this formalism in section two.

In section three we start from the axial-symmetric stationary field equations derived from equation (1) for the specific case $\alpha^2 = 3$. Using the formalism mentioned above, we will derive new solutions in section four, and finally show that the Kerr space-time, the Gibbons-Maeda [15], the Frolov-Zelnikov [17] and also their NUT generalizations are special cases of one of these new solutions.

2 The Potential-Space Formalism

For $\alpha^2 = 3$ the field equations can be derived from a five-dimensional space-time action. We will deal with a five-dimensional space-time possessing a Killing vector field X with close orbits. We will work with stationary space-times, this symmetry implies the existence of a second Killing vector field Y with close orbits as well. Thus, we start with a five-dimensional space-times possessing two commuting Killing vectors fields; a space-like one X , representing the inner symmetry and a time-like one Y , representing stationarity. The potential formalism consists in defining covariantly five potentials in terms of the Killing vectors X and Y . The five potentials are given by [16]

$$\begin{aligned} I^2 &= \kappa^{4/3} = X_A X^A; & f &= -I Y_A Y^A + I^{-1} (X^A Y_A)^2 \\ \psi &= -I^{-2} X_A Y^A; & \epsilon_{,A} &= \epsilon_{ABCDE} X^B Y^B X^{D;E} \\ \chi_{,\alpha} &= -\epsilon_{ABCDE} X^B Y^B X^{D;E} \end{aligned} \tag{12}$$

($A, B, \dots = 1 \dots 5$), where f, ϵ, ψ, χ and κ respectively are the gravitational, rotational, electrostatic, magnetostatic and scalar potentials; ϵ_{ABCDE} is the five-dimensional Levi-Civita pseudo-tensor, $X = X^A \partial/\partial x^A = \partial/\partial x^5$, and $Y = Y^A \partial/\partial x^A = \partial/\partial t$. We will work with spaces possessing axial symmetry as well, which is a realistic assumption for a star. Thus for the axial symmetric stationary case we have another Killing vector $Z = Z^A \partial/\partial x^A = \partial/\partial \varphi$, representing this symmetry. The field equations (2) in terms of the five po-

tentials $\Psi^A = (f, \epsilon, \psi, \chi, \kappa)$ read [18, 11]

$$\begin{aligned}
\hat{D}^2 \kappa + \left(\frac{\hat{D}\rho}{\rho} - \frac{\hat{D}\kappa}{\kappa} \right) \hat{D}\kappa + \frac{3\kappa^3}{4f} (\hat{D}\psi^2 - \frac{1}{\kappa^4} \hat{D}\chi^2) &= 0, \\
\hat{D}^2 \psi + \left(\frac{\hat{D}\rho}{\rho} + \frac{2\hat{D}\kappa}{\kappa} - \frac{\hat{D}f}{f} \right) \hat{D}\psi - \frac{1}{\kappa^2 f} (\hat{D}\epsilon - \psi \hat{D}\chi) \hat{D}\chi &= 0, \\
\hat{D}^2 \chi + \left(\frac{\hat{D}\rho}{\rho} - \frac{2\hat{D}\kappa}{\kappa} - \frac{\hat{D}f}{f} \right) \hat{D}\chi + \frac{\kappa^2}{f} (\hat{D}\epsilon - \psi \hat{D}\chi) \hat{D}\psi &= 0, \\
\hat{D}^2 f + \left(\frac{\hat{D}\rho}{\rho} - \frac{\hat{D}f}{f} \right) \hat{D}f + \frac{1}{f} (\hat{D}\epsilon - \psi \hat{D}\chi)^2 - \frac{\kappa^2}{2} \left(\hat{D}\psi^2 + \frac{1}{\kappa^4} \hat{D}\chi^2 \right) &= 0, \\
\hat{D}^2 \epsilon - \hat{D}\psi \hat{D}\chi - \psi \hat{D}^2 \chi + \left(\frac{\hat{D}\rho}{\rho} - \frac{2\hat{D}f}{f} \right) (\hat{D}\epsilon - \psi \hat{D}\chi) &= (13)
\end{aligned}$$

where \hat{D} is the differential operator $\hat{D} = (\partial_\rho, \partial_\zeta)$. The field equations (13) can be derived from the Lagrangian [16, 12, 11]

$$\mathcal{L} = \frac{\rho}{2f^2} [f_{,i}f^{,i} + (\epsilon_{,i} - \psi\chi_{,i})(\epsilon^{,i} - \psi\chi^{,i})] + \frac{\rho}{2f} \left(\kappa^2 \psi_{,i} \psi^{,i} + \frac{1}{\kappa^2} \chi_{,i} \chi^{,i} \right) - \frac{2\rho}{3\kappa^2} \kappa_{,i} \kappa^{,i}, \quad (14)$$

with $i = (\rho, \zeta)$. The next step is to look for the invariant transformations of Lagrangian (14), which were found in [19]. The invariance group of the Lagrangian (14) is $SL(3, \mathbb{R})$. We can write these transformations in a very simple form as

$$h \rightarrow Ch_0C^T, \quad (15)$$

where h and C are elements of $SL(3, \mathbb{R})$. One parameterization of matrix h is given by [18, 12]

$$h = -\frac{1}{f\kappa^{2/3}} \begin{pmatrix} f^2 + \epsilon^2 - f\kappa^2\psi^2 & -\epsilon & -\epsilon\chi + f\kappa^2\psi \\ -\epsilon & 1 & \chi \\ -\epsilon\chi + f\kappa^2\psi & \chi & \chi^2 - \kappa^2 f \end{pmatrix} \quad (16)$$

In terms of matrix h , it is possible to write down the field equations (13) in a non-linear σ -model form

$$(\rho h_{,z} h^{-1})_{,\bar{z}} + (\rho h_{,\bar{z}} h^{-1})_{,z} = 0. \quad (17)$$

Thus, we can define an abstract Riemannian space using the standard metric of the group defined by

$$\begin{aligned} ds^2 &= \frac{1}{4} \text{tr}(dh dh^{-1}) \\ &= \frac{\rho}{2f^2} [d f^2 + (d\epsilon - \psi d\chi)^2] - \frac{\rho \kappa^2}{2f} (d\psi^2 + \frac{1}{\kappa^4} d\chi^2) + \frac{2\rho}{3\kappa^2} d\kappa^2 \end{aligned} \quad (18)$$

This Riemannian space defines a five-dimensional symmetric space (the covariant derivative of the Riemannian tensor vanishes), with a isometry group $SL(3, \mathbb{R})$.

In what follows we will write explicitly the potentials Ψ^A in terms of the metric components. In order to do so, we recall that the five-dimensional space-time metric in terms of the four-dimensional one and the electromagnetic and scalar fields reads

$$ds_5^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu + I^2 (A_\mu dx^\mu + dx^5) (A_\nu dx^\nu + dx^5), \quad (19)$$

where $\hat{g}_{\mu\nu}$; $\mu, \nu = 1, \dots, 4$ are the 4-dimensional metric components of the five-dimensional space-time, I is the scalar potential and A_μ is the electromagnetic four potential. For the axial symmetric stationary case I, A_μ and $\hat{g}_{\mu\nu}$ depend only on ρ and ζ . The five-dimensional metric and its inverse can be written as

$$\hat{g}_{AB} = \begin{pmatrix} \hat{g}_{\mu\nu} + I^2 A_\mu A_\nu & I^2 A_\mu \\ I^2 A_\nu & I^2 \end{pmatrix} \quad (20)$$

$$\hat{g}^{AB} = \begin{pmatrix} \hat{g}^{\nu\tau} & -A^\nu \\ A^\nu & A^2 + I^{-3} \end{pmatrix} \quad (21)$$

Due to the symmetries we are working with, its is convenient to write the four-dimensional metric in the Papapetrou parameterization

$$ds_4^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{1}{f} (e^{2k} dz d\bar{z} + \rho^2 d\varphi^2) - f(\omega d\varphi + dt)^2 \quad (22)$$

In terms of this parameterization, and recalling that $ds_5^2 = \hat{g}_{AB} dx^A dx^B = \frac{1}{I} ds_4^2 + I^2 (A_\mu dx^\mu + dx^5)^2$, the metric coefficients \hat{g}_{AB} can be written as

$$\hat{g}_{AB} = \begin{pmatrix} 0 & \frac{1}{2If} e^{2k} & 0 & 0 & 0 \\ \frac{1}{2If} e^{2k} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\rho^2}{I} - \frac{f\omega^2}{I} + I^2 A_3^2 & \frac{f\omega}{I} + I^2 A_3 A_4 & I^2 A_3 \\ 0 & 0 & \frac{f\omega}{I} + I^2 A_3 A_4 & -\frac{f}{I} + I^2 A_4^2 & I^2 A_4 \\ 0 & 0 & I^2 A_3 & I^2 A_4 & I^2 \end{pmatrix} \quad (23)$$

Using the expressions for the metric given in equation (22), it is straightforward to calculate the gravitational, electrostatic and scalar potentials. Recalling that the Killing vectors components X^A and Y^A satisfy the relations $X^A = \delta^A{}_5$ and $Y^A = \delta^A{}_4$, one finds that $Y^A Y_A = \hat{g}_{44}$; and $X^A Y_A = \hat{g}_{54}$. Now, substituting these relations into the definition for $f = -I\hat{g}_{44} + I^{-1}(\hat{g}_{54})^2$ and using the relations (23) we obtain

$$f = -Ig_{44} \quad (24)$$

In similar way one obtains for the electrostatic and the scalar potentials:

$$\begin{aligned} \kappa^{\frac{4}{3}} &= \hat{g}_{55} = I^2, \\ \psi &= -A_4. \end{aligned} \quad (25)$$

For the magnetostatic and rotational potentials the corresponding expressions can be reduced to

$$\begin{aligned} \epsilon_{,\mu} &= \epsilon_{45\gamma\delta\mu} \hat{g}^{\tau\delta} \hat{g}^{\gamma\theta} \hat{g}_{4\theta,\tau}, \\ \chi_{,\mu} &= -\epsilon_{54\gamma\delta\mu} \hat{g}^{\tau\delta} \hat{g}^{\gamma\theta} \hat{g}_{5\theta,\tau}. \end{aligned} \quad (26)$$

Using now the relations (20), and (12) in (26), we find:

$$\begin{aligned} \epsilon_{,\bar{z}} &= -\frac{I^2}{\rho} [(g_{34}g_{44,\bar{z}} - g_{44}g_{34,\bar{z}})] + \psi\chi_{,\bar{z}}, \\ \epsilon_{,z} &= \frac{I^2}{\rho} [(g_{34}g_{44,z} - g_{44}g_{34,z})] + \psi\chi_z, \\ \chi_{,\bar{z}} &= \frac{I^4}{\rho} (g_{34}A_{4,\bar{z}} - g_{44}A_{3,\bar{z}}), \\ \chi_{,z} &= -\frac{I^4}{\rho} (g_{34}A_{4,z} - g_{44}A_{3,z}). \end{aligned}$$

The potentials are written in terms of g_{34} and g_{44} components of the four-dimensional metric tensor as well as of the A_3 and A_4 components of the electromagnetic four potential. From (26) and (23) we arrive at the final expressions

$$A_{3,z} = -\frac{\rho}{f\kappa^2} \chi_{,z} + \frac{g_{34}I}{f} \psi_{,z}$$

$$\begin{aligned}
A_{3,\bar{z}} &= -\frac{\rho}{f\kappa^2}\chi_{,\bar{z}} + \frac{g_{34}I}{f}\psi_{\bar{z}} \\
\left(\frac{g_{34}}{g_{44}}\right)_{,z} &= \frac{\epsilon_{,z} - \psi\chi_{,z}}{f^2}
\end{aligned} \tag{27}$$

In the following sections we will use these expressions for obtaining exact solutions of the field equations.

3 Calculations and Solutions

In this section we will apply the previous results for finding exact solutions of the field equations. Let us consider f_0, I_0, ψ_0, \dots , etc. as seed solutions, *i.e.*, as components of the matrix h_0 in (15). We proceed as follows. First, using the inverse matrix h^{-1} , and the $SL(3, \mathbb{R})$ invariance of the field equations, from (15) we obtain the h components in terms of the Ψ_0^A potentials. Finally, using a particular matrix C in (15) we integrate the new potentials Ψ^A in general for this particular cases.

The inverse matrix of (16) reads

$$h^{-1} = -\frac{\kappa^{2/3}}{f} \begin{pmatrix} 1 & \epsilon - \chi\psi & \psi \\ \epsilon - \chi\psi & f^2 + (\epsilon - \chi\psi)^2 - f\chi^2\kappa^{-2} & \kappa^{-2}f\chi + \psi(\epsilon - \chi\psi) \\ \psi & \chi f\kappa^{-2} + \psi(\epsilon - \chi\psi) & -\kappa^{-2}f + \psi^2 \end{pmatrix}. \tag{28}$$

Then, we can write the potentials Ψ^A in terms of the components of matrices h and h^{-1}

$$\begin{aligned}
\kappa^{\frac{4}{3}} &= \frac{h_{11}^{-1}}{h_{22}}; & f^2 &= \frac{1}{h_{11}^{-1}h_{22}}; & \chi &= \frac{h_{23}}{h_{22}} \\
\psi &= \frac{h_{13}^{-1}}{h_{11}^{-1}}; & \epsilon &= -\frac{h_{12}}{h_{22}}
\end{aligned} \tag{29}$$

where we have used the notation $h_{ij}^{-1} = (h^{-1})_{ij}$. Using expressions (29) we can straightforwardly calculate the potentials Ψ^A from the h components. We will take each case separately.

3.1 Case $\Psi_0^A = (f_0, 0, \psi_0, 0, \kappa_0)$

In this case we start from a solution with electrostatic, scalar and gravitational potentials, for this case matrix h_0 reads

$$h_0 = -\frac{1}{\kappa_0^{2/3} f_0} \begin{pmatrix} f_0^2 - f_0 \kappa_0^2 \psi_0 & 0 & f_0 \kappa_0^2 \psi_0 \\ 0 & 1 & 0 \\ f_0 \kappa_0^2 \chi_0 & 0 & -\kappa_0^{-2} f_0 \end{pmatrix} \quad (30)$$

The inverse matrix is given by

$$h_0^{-1} = -\frac{\kappa_0^{2/3}}{f_0} \begin{pmatrix} 1 & 0 & \psi_0 \\ 0 & f_0^2 & 0 \\ \psi_0 & 0 & -\kappa_0^{-2} f_0 + \psi_0^2 \end{pmatrix} \quad (31)$$

Now we take the invariance equation $h = ChC^T$ taking the constant matrix C arbitrary as

$$C = \begin{pmatrix} a & b & c \\ d & e & j \\ i & h & k \end{pmatrix} \quad (32)$$

and its inverse as

$$C^{-1} = \begin{pmatrix} q & p & t \\ u & v & w \\ s & y & z \end{pmatrix}. \quad (33)$$

Substituting it into (29) we arrive at

$$\begin{aligned} \kappa^{\frac{4}{3}} &= \frac{U}{V} \\ f^2 &= \frac{f_0^2 \kappa_0^{\frac{8}{3}}}{UV} \\ \chi &= \frac{idf_0^2 - (id - ij)f_0 \kappa_0^2 \psi_0^2 + (dk\psi_0 - kj)f_0 \kappa_0^2 + eh}{V \kappa_0^{-\frac{2}{3}}} \\ \psi &= -\frac{\kappa_0^2 [-tq - (ts + zq)\psi_0 - wuf_0^2 - sz\psi_0^2] + szf_0}{U} \\ \epsilon &= -\frac{f_0 [da f_0 + \kappa_0^2 \psi_0^2 (dc - da) + \kappa_0^2 (ja\psi_0 - jc)] + be}{V \kappa_0^{-\frac{2}{3}}} \end{aligned} \quad (34)$$

with $U = \kappa_0^2(q^2 + 2qs\psi_0 + u^2f_0^2 + s^2\psi_0^2) - s^2f_0$ and $V = \kappa_0^{\frac{2}{3}}[df_0^2 - \kappa_0^2(f_0\psi_0^2 - 2djf_0\psi_0 + j^2f_0) + e^2]$. In order to perform a total integration of the metric components, we take the matrix C as

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & v & -w \\ 0 & -w & v \end{pmatrix} \quad (35)$$

with inverse

$$C^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & v & w \\ 0 & w & v \end{pmatrix} \quad (36)$$

Then equations (34) reduce to the simple expressions

$$\begin{aligned} \kappa^{\frac{4}{3}} &= \frac{\kappa_0^{\frac{4}{3}}}{v^2 - w^2\kappa_0^2f_0} \\ f^2 &= \frac{f_0^2}{v^2 - w^2\kappa_0^2f_0} \\ \chi &= -\frac{vw(1 - f_0\kappa_0^2)}{v^2 - w^2\kappa_0^2f_0} \\ \psi &= v\psi_0 \\ \epsilon &= \frac{wf_0\psi_0\kappa_0^2}{v^2 - w^2\kappa_0^2f_0} \end{aligned} \quad (37)$$

keeping in mind that matrix C fulfills the condition $\det C = 1$, *i.e.* $v^2 - w^2 = 1$, from equations (37), we obtain the following relation:

$$\frac{\epsilon_{,l} - \psi\chi_{,l}}{f^2} = \frac{w\kappa_0^2\psi_{0,l}}{f_0}, \quad (38)$$

with $l = z, \bar{z}$. Then, from these last expressions (38), and from the last pair of equations (27), using equation (37), we have that

$$\left(\frac{g_{34}I}{f} \right)_{,z} = -\frac{w\rho\kappa_0^2\psi_{0,z}}{f_0}; \quad \left(\frac{g_{34}I}{f} \right)_{,\bar{z}} = \frac{w\rho\kappa_0^2\psi_{0,\bar{z}}}{f_0}. \quad (39)$$

We start using the solution (6) and (7), with $\alpha^2 = 3$ as seed solution and substituting it into equations (37) to obtain

$$f^2 = \frac{e^{2\lambda}}{(a_1\tau + a_2)^{1/2}(v^2 - w^2k_1^2(a_1\tau + a_2)e^{2\lambda})}$$

$$\begin{aligned}
\chi &= -\frac{vw(1-k_1^2(a_1\tau+a_2)e^{2\lambda})}{v^2-w^2k_1^2(a_1\tau+a_2)e^{2\lambda}} \\
\psi &= v\frac{a_3\tau+a_4}{a_1\tau+a_2} \\
\epsilon &= \frac{wk_1^2(a_3\tau+a_4)}{v^2-w^2k_1^2(a_1\tau+a_2)e^{2\lambda}} \\
\kappa^{4/3} &= \kappa_0^{4/3}\frac{(a_1\tau+a_2)e^{2/3\lambda}}{v^2-w^2k_1^2(a_1\tau+a_2)e^{2\lambda}}
\end{aligned} \tag{40}$$

and substituting this seed solution (6) and (7) together with the restriction (8) into expressions (39), we arrive at:

$$\left(\frac{g_{34}I}{f}\right)_{,z} = -wa_1k_1\rho\tau_{,z}; \quad \left(\frac{g_{34}I}{f}\right)_{,\bar{z}} = wa_1k_1\rho\tau_{,\bar{z}}. \tag{41}$$

The integrability of the right hand side of expression (41) is guaranteed because τ is harmonic and fulfills the Laplace equation (4). The explicitly form of the function g_{34} depends on τ . In [20][21] is presented a list of expressions of the rhs of (39) for different τ . In terms of g_{34} and the solution given in (40), we can write the final metric (19) as

$$\begin{aligned}
ds_5^2 &= \frac{B}{\kappa_1^{\frac{2}{3}}}e^{2\kappa-\frac{4\lambda}{3}}dzd\bar{z} + \left(\rho^2e^{-\lambda}m_1^{\frac{1}{2}} - \frac{\kappa_1^{\frac{2}{3}}m_1g_{34}^2}{e^{\frac{2\lambda}{3}}}\right)\frac{e^{\frac{4\lambda}{3}}}{\kappa_1^{\frac{4}{3}}m_1^2}d\varphi^2 + 2g_{34}d\varphi dt \\
&- \frac{e^{\frac{2\lambda}{3}}}{\kappa_1^{\frac{2}{3}}m_1}dt^2 + \kappa_1m_1^{\frac{1}{2}}e^{\frac{\lambda}{3}}(A_3d\varphi - v\frac{a_3\tau+a_4}{m_1}dt + dx^5)^2
\end{aligned} \tag{42}$$

with

$$A_{3,z} = \frac{\rho}{w\kappa_1^{\frac{2}{3}}e^{\frac{4\lambda}{3}}}(\ln B)_{,z} + \frac{ve^{\frac{2\lambda}{3}}}{\kappa_1^{\frac{5}{3}}m_1}g_{34}\tau_{,z} \tag{43}$$

and $B = v^2 - w^2e^{2\lambda}\kappa_1m_1$, $m_1 = a_1\tau + a_2$. That is, for a given τ , we are able to obtain a rotating exact solution, with scalar, magnetic and gravitational fields using formulas (39) and (43).

For the second subclass we use the solution (6) with (10) with $\alpha^2 = 3$ into (37) to obtain

$$f^2 = \frac{e^{2\lambda}}{(a_1e^{q_1\tau} + a_2e^{q_2\tau})(v^2 - w^2k_1^2(a_1e^{q_1\tau} + a_2e^{q_2\tau})e^{2\lambda+\tau_0\tau})}$$

$$\begin{aligned}
\chi &= -vw \frac{1 - k_1^2(a_1 e^{q_1 \tau} + a_2 e^{q_2 \tau}) e^{2\lambda + \tau_0 \tau}}{v^2 - w^2 k_1^2(a_1 e^{q_1 \tau} + a_2 e^{q_2 \tau}) e^{2\lambda + \tau_0 \tau}} \\
\psi &= v \frac{a_3 e^{q_1 \tau} + a_4 e^{q_2 \tau}}{a_1 e^{q_1 \tau} + a_2 e^{q_2 \tau}} \\
\epsilon &= \frac{w k_1^2(a_3 e^{q_1 \tau} + a_4 e^{q_2 \tau}) e^{2\lambda + \tau_0 \tau}}{v^2 - w^2 k_1^2(a_1 e^{q_1 \tau} + a_2 e^{q_2 \tau}) e^{2\lambda + \tau_0 \tau}} \\
\kappa^{4/3} &= \kappa_0^{4/3} \frac{(a_1 e^{q_1 \tau} + a_2 e^{q_2 \tau}) e^{2/3(\lambda + \tau_0 \tau)}}{v^2 - w^2 k_1^2(a_1 e^{q_1 \tau} + a_2 e^{q_2 \tau}) e^{2\lambda + \tau_0 \tau}}
\end{aligned} \tag{44}$$

In this case, relations (39) with the seed solution (6) with (10) and the constraints among the constant, equation(9), give us

$$\begin{aligned}
\left(\frac{g_{34} I}{f} \right)_{,z} &= -\frac{w(q_1 - q_2)a_1 \kappa_1}{\sqrt{a_1 a_2}} \rho \tau_{,z}; \\
\left(\frac{g_{34} I}{f} \right)_{,\bar{z}} &= \frac{w(q_1 - q_2)a_1 \kappa_1}{\sqrt{a_1 a_2}} \rho \tau_{,\bar{z}}.
\end{aligned} \tag{45}$$

As in the last case, we can now obtain the line element in terms of the function g_{34} , which can be integrated from these last pair of equations once a τ is given. The metric then reads

$$\begin{aligned}
ds_5^2 &= \frac{B}{\kappa_1^{\frac{2}{3}} m_1^{\frac{1}{2}}} e^{2\kappa - (\frac{4\lambda}{3} + (q_1 + q_2)\tau)} dz d\bar{z} + \left(\rho^2 e^{-\lambda} m_1^{\frac{1}{2}} - \frac{\kappa_1^{\frac{2}{3}} m_1 g_{34}^2}{e^{\frac{\lambda}{3} + (q_1 + q_2)\tau}} \right) \times \\
&\quad \frac{e^{2\frac{\lambda}{3} + 2(q_1 + q_2)\tau}}{\kappa_1^{\frac{4}{3}} m_1^2} d\varphi^2 + 2g_{34} d\varphi dt - \frac{e^{\frac{\lambda}{3} + (q_1 + q_2)\tau}}{\kappa_1^{\frac{2}{3}} m_1^{\frac{3}{2}}} dt^2 \\
&\quad + \kappa_1^{\frac{2}{3}} m_1 e^{\frac{2\lambda}{3}} (A_3 d\varphi - v \frac{a_3 e^{q_1 \tau} + a_4 e^{q_2 \tau}}{m_1} dt + dx^5)^2.
\end{aligned} \tag{46}$$

The function A_3 again depends on the harmonic function τ as

$$A_{3,z} = \frac{\rho}{\omega \kappa_1^{\frac{2}{3}} e^{\frac{4\lambda}{3}}} (\ln B)_{,z} + \frac{v a_1 (q_1 - q_2) e^{\frac{\lambda}{3} + 2(q_1 + q_2)\tau}}{\kappa_1^{\frac{5}{3}} m_1} g_{34} \tau_{,z} \tag{47}$$

where $B = v^2 - \omega^2 e^{2\lambda + (q_1 + q_2)\tau} \kappa_1 m_1$ and $m_1 = a_1 e^{q_1 \tau} + a_2 e^{q_2 \tau}$. With solutions (42) and (46) we are now able to obtain rotating exact solutions which represent rotating monopoles, dipoles, etc. coupled to a dilaton field.

3.2 Case $\Psi_0^A = (f_0, 0, 0, \chi_0, \kappa_0)$

We start from a static solution with magnetostatic, scalar and gravitational potentials. In this case, the matrix h_0 is

$$h_0 = -\frac{1}{\kappa_0^{2/3} f_0} \begin{pmatrix} f_0^2 & 0 & 0 \\ 0 & 1 & \chi_0 \\ 0 & \chi_0 & -\chi_0^2 - \kappa_0^{-2} f_0 \end{pmatrix} \quad (48)$$

with inverse

$$h_0^{-1} = -\frac{\kappa_0^{2/3}}{f_0} \begin{pmatrix} 1 & 0 & 0 \\ 0 & f_0^2 + \chi_0^2 \kappa_0^{-2} f_0 + f_0^2 & f_0 \chi_0 \kappa_0^{-2} \\ 0 & f_0 \chi_0 \kappa_0^{-2} & -f_0 \kappa_0^{-2} \end{pmatrix}. \quad (49)$$

Using the invariance equation (15) and substituting expressions (48) and (49) into the set given by equation (29), with the matrix C given by equation (32), we obtain:

$$\begin{aligned} \kappa_0^{\frac{4}{3}} &= \frac{1}{\kappa_0^{2/3}} \frac{B}{A} \\ f^2 &= \frac{f_0^2 \kappa_0^2}{AB} \\ \chi &= \frac{1}{A} \{d f_0^2 + e(h + k \chi_0) + j(h \chi_0 + k(\chi_0^2 - f_0 \kappa_0^2))\} \\ \psi &= \frac{1}{B} \{q t \kappa_0^2 + u w(-\chi_0^2 f_0 + \kappa_0^2 f_0^2) + \chi_0 f_0(s w + u z) - s z f_0\} \\ \epsilon &= -\frac{1}{A} \{a d f_0^2 + b e + \chi_0(b j + c e) + c j(\chi_0^2 - \kappa_0^2 f_0)\} \end{aligned} \quad (50)$$

with $A = d^2 f_0^2 + e(e + 2 j \chi_0) + j^2(\chi_0^2 - \kappa_0 f_0)$ and $B = q^2 \kappa_0^2 + u^2(f_0^2 \kappa_0^2 - f_0 \chi_0^2) + s f_0(2 u \chi_0 - s)$.

Next, we take for the matrix C the following particular form:

$$C = \begin{pmatrix} q & 0 & -s \\ 0 & 1 & 0 \\ -s & 0 & q \end{pmatrix}, \quad (51)$$

and its inverse

$$C^{-1} = \begin{pmatrix} q & 0 & s \\ 0 & 1 & 0 \\ s & 0 & q \end{pmatrix}, \quad (52)$$

thus $q^2 - s^2 = 1$. With this particular form of C , the potentials reduce to the following expressions:

$$\begin{aligned}\kappa^{\frac{4}{3}} &= \kappa_0^{4/3}(q^2 - s^2 f_0 \kappa_0^{-2}) \\ f &= \frac{f_0}{\sqrt{q^2 - s^2 f_0 \kappa_0^{-2}}} \\ \chi &= q\chi_0 \\ \epsilon &= s\chi_0 \\ \psi &= \frac{sq(1 - f_0 \kappa_0^{-2})}{q^2 - s^2 f_0 \kappa_0^{-2}}.\end{aligned}\tag{53}$$

Using the first differential equation of expressions (27) for A_3 in terms of the seed potentials, (for this case $\epsilon_0 = \psi_0 = 0$), we have that

$$\begin{aligned}A_{03,z} &= -\frac{\rho}{f_0 I_0^3} \chi_{0,z} \\ A_{03,\bar{z}} &= \frac{\rho}{f_0 I_0^3} \chi_{0,\bar{z}}.\end{aligned}\tag{54}$$

On the other hand, from the potentials given in (53), recalling that $\kappa_0^2 = I_0^3$, we obtain that in this case:

$$\frac{\epsilon_{,z} - \psi \chi_{,z}}{f^2} = \frac{s}{f_0 I_0^3} \chi_{0,z}.\tag{55}$$

Thus, using the last differential equation from (27) we find that

$$-\frac{1}{\rho} \left(\frac{g_{34}}{g_{44}} \right)_{,z} = \frac{s}{f_0 I_0^3} \chi_{0,z}.\tag{56}$$

From equation (54) and this last one, we obtain that

$$\left(\frac{g_{34}}{g_{44}} \right)_{,z} = s A_{03,z},\tag{57}$$

which implies that (up to a constant)

$$\left(\frac{g_{34}}{g_{44}} \right) = s A_{03},\tag{58}$$

that is

$$g_{34} = g_{44} s A_{03} = \frac{f}{I} s A_{03}, \quad (59)$$

i.e., for this case we do not need to perform any extra integration for generating a new rotating solution starting from the seed one. In this way, we finally obtain the following expression for the target metric

$$\begin{aligned} ds_5^2 = & \frac{1}{I} \left\{ T^{\frac{1}{2}} \frac{e^{k_0}}{f_0} dz d\bar{z} + \left[T^{\frac{1}{2}} \frac{\rho^2}{f_0} - \frac{s^2 A_{03}^2 f_0}{T^{\frac{1}{2}}} \right] d\varphi^2 - \frac{s A_{03} f_0}{T^{\frac{1}{2}}} d\varphi dt - \frac{f_0}{T^{\frac{1}{2}}} dt^2 \right\} \\ & + I^2 \left(\frac{q A_{03}}{T} d\varphi - \frac{q s (1 - f_0 \kappa_0^{-2})}{T} dt + dx^5 \right)^2 \end{aligned} \quad (60)$$

where $T = q^2 - s^2 f_0 \kappa_0^{-2}$, $A_3 = q A_{03} / (q^2 - s^2 f_0 \kappa_0^{-2})$ and $A_4 = q s (1 - s^2 f_0 \kappa_0^{-2}) / (q^2 - s^2 f_0 \kappa_0^{-2})$. Thus, we have generated again a new exact solution to the Einstein-Maxwell-dilaton theory, for $\alpha^2 = 3$, in which all the fields are non trivially involved.

3.3 case $\Psi_0^A = (f_0, \epsilon_0, 0, 0, \kappa_0)$

This case was studied in [9], we presented it here somewhat more detailed in order to have all the cases together. In this case we take as initial solution one without electrostatic and magnetostatic fields. The matrix h_0 is

$$h_0 = -\frac{1}{\kappa^{2/3} f_0} \begin{pmatrix} f_0^2 + \epsilon_0^2 & -\epsilon_0 & 0 \\ -\epsilon_0 & 1 & 0 \\ 0 & 0 & -\kappa_0^{-2} f_0 \end{pmatrix} \quad (61)$$

and its inverse is

$$h_0^{-1} = -\frac{\kappa^{2/3}}{f_0} \begin{pmatrix} 1 & \epsilon_0 & 0 \\ \epsilon_0 & f_0^2 + \epsilon_0^2 & 0 \\ 0 & 0 & \frac{-f_0}{\kappa_0^2} \end{pmatrix}. \quad (62)$$

In a similar way as in the other cases presented in this work, we take the equations relating the components of the matrices given in equations (16) and (28), and using (32) and (33) in the invariance equation (15), we arrive at

$$\kappa^{\frac{4}{3}} = \frac{U}{\kappa_0^{\frac{2}{3}} V}$$

$$\begin{aligned}
f^2 &= \frac{f_0^2 \kappa_0^2}{UV} \\
\chi &= \frac{idf_0^2 + \epsilon_0(id\epsilon_0 - ie - hd) + he - j\kappa\kappa_0^2 f_0}{V\kappa_0^{\frac{-2}{V}}} \\
\psi &= \frac{\kappa_0^2[tq + wuf_0^2 + \epsilon_0(uw\epsilon_0 + tu + qw)] - szf_0}{U} \\
\epsilon &= -\frac{f_0(daf_0 - cj\kappa_0^2) + \epsilon_0(ad\epsilon_0 - bd - ae) + be}{V}
\end{aligned}$$

where $U = \kappa_0(q^2 + 2qu\epsilon_0 + (uf_0)^2 + (u\epsilon_0)^2) - s^2f_0$, and $V = (df_0)^2 + (d\epsilon_0)^2 - 2de\epsilon_0 + e^2 - (j\kappa_0)^2 f_0$. In order to integrate the metric, it is again necessary to consider a simpler matrix C . We take

$$C = \begin{pmatrix} q & 0 & -s \\ 0 & 1 & 0 \\ -s & 0 & q \end{pmatrix}, \quad (63)$$

and its inverse

$$C^{-1} = \begin{pmatrix} q & 0 & s \\ 0 & 1 & 0 \\ s & 0 & q \end{pmatrix}, \quad (64)$$

then, recalling that again $q^2 - s^2 = 1$, the potentials read

$$\begin{aligned}
\kappa^{\frac{4}{3}} &= B\kappa_0^{\frac{4}{3}} \\
f^2 &= f_0^2 B^{-1} \\
\chi &= s\epsilon_0 \\
\psi &= \frac{sq[1 - \kappa_0^{-2} f_0]}{B} \\
\epsilon &= q\epsilon_0
\end{aligned} \quad (65)$$

where $B = q^2 - s^2\kappa_0^{-2}f_0$. Integrating equation (27), and substituting in it expression (65), we obtain:

$$\frac{1}{\rho} \left(\frac{g_{34}I}{f} \right)_{,z} = \frac{q}{\rho} \left(\frac{g_{034}I_0}{f_0} \right)_{,z}, \quad (66)$$

which implies that the expression $\frac{1}{\rho} \left(\frac{g_{34}I}{f} \right)_{,z}$ remains invariant (up to a constant) for the seed solution and the generated one, *i.e.*

$$g_{34} = \frac{q}{B} g_{034}. \quad (67)$$

Similarly we find that

$$A_3 = -s \frac{g_{034} I_0}{B}, \quad (68)$$

i.e., again, for this case we do not need to perform any extra integration for generating a new rotating solution starting from the seed one. Substituting the solution into the Papapetrou metric (19) we arrive at

$$\begin{aligned} ds_5^2 = & \frac{1}{I_0 f_0} e^{2k} dz d\bar{z} - B \left[g_{033} + \frac{g_{034}^2 I_0}{f_0} \left(1 - \frac{q^2}{B} \right) \right] d\varphi^2 + 2 \frac{q}{B} g_{034} d\varphi dt \\ & - \frac{f_0}{B I_0} dt^2 + I^2 \left(\frac{-s g_{034} I_0}{\kappa_0^2 B} d\varphi - \frac{s q [1 - \kappa_0^{-2} f_0]}{B} dt + dX^5 \right)^2 \end{aligned}$$

Here we must start from a static exact solution coupled to a scalar field. If there are no extra fields besides the scalar one, the Einstein equations decouple from the scalar one which satisfy a harmonic equation. The field equation for the scalar field can be integrated independently from the Einstein equations. As an exact solution for the scalar field equation we take the function $\kappa_0 = [(r - m + \sigma)/(r - m - \sigma)]^\delta$. For the Einstein equations we take as seed metric the Kerr-NUT space-time, *i.e.* as seed solution we have

$$\kappa_0 = \left(\frac{r - m + \sigma}{r - m - \sigma} \right)^\delta; \quad \epsilon_0 = \frac{2(\omega L_+ - lr)}{\omega}; \quad f_0 = \frac{\omega - 2mr - 2lL_+}{\omega}, \quad (69)$$

with

$$L_+ = a \cos \theta + l; \quad L_- = a \cos \theta - l; \quad \omega = r^2 + (a \cos \theta + l)^2 \quad (70)$$

where r and θ are the Boyer-Lindquist coordinates, $\rho = \sqrt{r^2 + 2mr + a^2 - l^2} \sin \theta$ and $\zeta = (r - m) \cos \theta$; a , m and l respectively are the rotation, mass and NUT parameters, σ , and δ are integration constants. The resulting target solution is an exact axial symmetric stationary solution of 5D gravity, with electromagnetic and scalar fields. It reads

$$ds_5^2 = \frac{\omega}{\omega - 2mr - 2lL_+} \left(\frac{r - m + \sigma}{r - m - \sigma} \right)^{\frac{2\delta}{3}} (r^2 - 2mr + L_+ L_-) e^{2k_s} \left(\frac{dr^2}{\Delta} + d\theta^2 \right)$$

$$\begin{aligned}
& + \frac{1}{D} \left(\frac{r-m+\sigma}{r-m-\sigma} \right)^{\frac{4\delta}{3}} \{ -(\omega - 2mr - 2lL_+) dt^2 \\
& - (4qa(mr+l) \sin^2 \theta - 4l \cos \theta \Delta) \frac{\omega - 2mr - 2lL_+}{r^2 - 2mr + L_+L_-} dt d\varphi \\
& + \left[\frac{\omega}{\omega - 2mr - 2lL_+} \Delta \sin^2 \theta D \left(\frac{r-m+\sigma}{r-m-\sigma} \right)^{2\delta} \right. \\
& \left. - q^2(\omega - 2mr - 2lL_+) \left(\frac{2a \sin^2 \theta (mr+l) + 2l \cos \theta \Delta}{r^2 - 2mr + L_+L_-} \right)^2 \right] d\varphi^2 \} \\
& + \left(\frac{r-m+\sigma}{r-m-\sigma} \right)^{\frac{2\delta}{3}} \frac{D}{\omega} (A_3 d\varphi + A_4 dt + dX^5)^2
\end{aligned} \tag{71}$$

where

$$A_3 = - \left(\frac{r-m+\sigma}{r-m-\sigma} \right)^{2\delta} \left(\frac{2a(mr+l) \sin^2 \theta + 2l \Delta \cos \theta}{r^2 - 2mr + L_+L_-} \right) \frac{\omega - 2mr + 2lL_+}{D} \tag{72}$$

$$A_4 = -qs \frac{\omega \left(\frac{r-m+\sigma}{r-m-\sigma} \right)^{2\delta} - (\omega - 2mr + 2lL_+)}{D} \tag{73}$$

$$\begin{aligned}
D &= \omega \left(\frac{r-m+\sigma}{r-m-\sigma} \right)^{2\delta} q^2 - s^2(\omega - 2mr + 2lL_+); \\
\Delta &= r^2 - 2mr + a^2 - l^2 \\
e^{2k_s} &= \left[\frac{(\sqrt{\rho^2 + (\zeta - m)^2} + \sqrt{\rho^2 + (\zeta + m)^2})^2 - 4m^2}{4\sqrt{(\rho^2 + (\zeta - m)^2)(\rho^2 + (\zeta + m)^2)}} \right]^{\frac{8}{3}\delta^2}
\end{aligned} \tag{74}$$

Exact solution (71) was first obtained in [9] (see also [22]) and contains a great amount of well-known solutions of 5D gravity. We will list three of the most important ones. In order to do so, we start setting $\delta = l = 0$ in (71), obtaining for the 4-dim space-time (see [9])

$$\begin{aligned}
ds_4^2 &= \frac{\sqrt{D\omega}}{\Delta} dr^2 + \sqrt{D\omega} d\theta^2 - \frac{\Delta - a^2 \sin^2 \theta}{\sqrt{D\omega}} dt^2 - \frac{4qamr \sin^2 \theta}{\sqrt{D\omega}} dt d\varphi \\
& + \frac{\sin^2 \theta}{\sqrt{D\omega}(\Delta - a^2 \sin^2 \theta)} [\Delta(D\omega) - 4q^2 a^2 m^2 r^2 \sin^2 \theta] d\varphi^2
\end{aligned} \tag{75}$$

where $D = \omega q^2 - s^2(\omega - 2mr)$; $\Delta = r^2 - 2mr + a^2$; $\omega = r^2 + a^2 \cos^2 \theta$. For the scalar field the solution reduces to

$$\kappa^{\frac{4}{3}} = q^2 - s^2 \left(1 - \frac{2mr}{r^2 + a^2 \cos^2 \theta} \right).$$

The following known solutions are contained as particular cases of the generated target solution whose ds_4^2 part is given by equation (22):

Frolov-Zelnikov Solution

This solution is a charged rotating black hole, obtained by Frolov and Zelnikov in 1987 [17]. The ds_4^2 part of the metric is given by

$$\begin{aligned} ds_4^2 = & -\frac{1-z}{B}dt^2 - 2a \sin^2 \theta \frac{1}{\sqrt{1-v^2}} \frac{z}{B} dt d\varphi \\ & + \left[B(r^2 + a^2) + a^2 \sin^2 \theta \frac{z}{B} \right] \sin^2 \theta d\varphi^2 + B \frac{\Sigma}{\Delta} dr^2 + B \Sigma d\theta^2, \end{aligned}$$

where $B = \sqrt{(1-v^2+v^2z)/(1-v^2)}$, $z = 2mr/\Sigma$, and $\Delta = r^2 + a^2 - 2mr$. Comparing it with equation (3.3), we find that this solution corresponds to $\delta = l = 0$, $a \neq 0$; $\Sigma = \omega$, $D = B^2\omega$ and $q = 1/(1-v^2)$.

Gibbons-Maeda-Horner-Horowitz Solution

This solution describes a dilatonic static charged black hole. If we set $a = 0$ in (75) we get the metric

$$ds_4^2 = \frac{A}{\Delta} dr^2 - \frac{\Delta}{A} dt^2 + r\sqrt{D}(d\theta^2 + \sin^2 \theta d\varphi^2) \quad (76)$$

where $\omega = r^2$; $D = \omega q^2 - s^2(r^2 - 2mr)$; $\Delta = r^2 - 2mr$; $A = \sqrt{D\omega}$. Metric (76) can be rewritten as

$$ds^2 = \frac{1}{f} dr^2 - f dt^2 + r\sqrt{D} d\Omega^2 \quad (77)$$

where

$$f = \frac{1 - \frac{2m}{r}}{\sqrt{q^2 - s^2 \frac{1-2m}{r}}}. \quad (78)$$

With the restriction $q^2 - s^2 = 1$, the function f transform into

$$f = \frac{1 - \frac{2m}{r}}{\sqrt{1 + \frac{2m}{r} s^2}} \quad (79)$$

If we set $r_+ - r_- = 2m$; $r_- = 2ms^2$; $r_+ = 2m(1 - s^2)$ we get $f = (1 + (r_- - r_+)/r)/\sqrt{1 + r_-/r}$; $r\sqrt{D} = R^2 = r^2\sqrt{1 + r_-/r}$ which corresponds just to the Gibbons-Maeda-Horner-Horowitz solution [15] [14]. The charge and the mass parameters can be written as $Q = ms\sqrt{1 - s^2}$; $M = m - \frac{1}{2}ms$. From the case $s = 0$ we find the Schwarzschild solution ($Q = 0$ and $M = m$).

Finally, we want to stress the fact that with the procedure of solution generation presented in this work, the seed solution is also included within the target solution as a particular case, thus we have:

Kerr solution

To recover the Kerr solution setting $\delta = l = 0$; $s = 0$, $q = 1$ and $a \neq 0$ in (75), we get

$$ds_4^2 = -\left(\frac{\Delta - a^2 \sin^2 \theta}{\omega}\right) dt^2 - \frac{2a \sin^2 \theta (r^2 + a^2 - \Delta)}{\omega} dt d\varphi + \sin^2 \theta d\varphi^2 + \frac{\omega}{\Delta} dr^2 + \omega d\varphi^2$$

with $\Delta = r^2 - 2mr + a^2$ and $\omega = r^2 + a^2 \cos^2 \theta$.

NUT parameter

We can also obtain the NUT solution [23] taking $a = 0$, $\delta = 0$ and $l \neq 0$

$$ds_4^2 = -\frac{r^2 + l^2}{\Delta^2} dr^2 + (r^2 + l^2) d\theta^2 + (r^2 + l^2) \sin^2 \theta d\varphi^2 - \frac{\Delta}{r^2 + l^2} dt^2$$

$\Delta = r^2 - 2mr - l^2$ where l and m respectively are the NUT parameter and the mass.

4 Conclusions

In this work we have given a series of formulae to obtain rotating exact solutions of the Einstein-Maxwell-Dilaton field equations generated from seed static ones. The examples we gave for the application of these formulae, consist on start from a seed solutions in terms of harmonic maps, *i.e.*, in terms of two functions which fulfill the Laplace equation. The static seed solutions represent gravitational fields coupled to a scalar field and to a magnetostatic (electrostatic) monopoles, dipoles, quadrupoles, etc. The new solutions generated using our formulae represent the rotating version of the seed ones. The

new solutions contain induced electric (magnetic) fields generated by the rotation of the body. Some of the seed solutions model the exterior field of a pulsar containing a scalar field in the slow rotation limit. The scalar fields could be fundamental or generated by spontaneous scalarization [5]. The new rotating version of the solution generated using our formulae are in this sense more realistic and could represent the exterior field of a pulsar with fast rotation. We suggest that this solutions could be used as theoretical models for testing the strong gravitational regime of the Einstein theory or the most important generalizations of general relativity near a pulsar containing a scalar field. Because of the presence of the electromagnetic field, our solutions could give also a light to the understanding of such strong effects like the origin of jets and maybe of the origin of the QPOs, where approximated and numerical methods could be not completely trustable.

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